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CENTRE DE RECHERCHE SUR LES TRANSPORTS

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Patrice Marcotte

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DETERMINATION OF OLIGOPOLISTIC EQUILIBRIA: THE NONDIFFERENTIABLE CASE*

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ABSTRACT: Solutions to the oligopolistic equilibrium problem have been obtained under a variety of concavity and differentiability assumptions about the supply and inverse demand curves. In this paper we devise a search procedure to find all equilibrium points. Differentiability of the functions involved is not required. We show that, even under strict concavity assumptions, the equilibrium is not necessarily unique. Finally we describe a context of derived demand where nondifferentiability occurs naturally, as well as a numerical comparison with another algorithm.

RESUME: Dans cet article, nous présentons un algorithme de calcul des états d'équilibre d'un oligopole qui n'exige pas la différentiabilité des courbes d'offre et de demande. Nous montrons que, même sous des hypothèses de stricte concavité, l'équilibre n'est pas nécessairement unique. Finalement, nous décrivons une situation où la non différentiabilité intervient "naturellement", ainsi qu'une comparaison numérique avec un autre algorithme.

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** GERAD (Groupe d'Etudes et de Recherche en Analyse des Décisions), Ecole des HEC, 5255 avenue Decelles, Montréal H3T 1V6, Québec, Canada.

1. INTRODUCTION

Algorithms for finding Nash equilibria follow basically two lines of attack; the first uses the fixed point approach of Scarf [5], the second is based on more standard mathematical programming methods and is better adapted to the solution of large scale problems ([4], [6]). Following this latter approach, Murphy et al [4] proposed an efficient algorithm based on replacing the original problem by a sequence of constrained concave maximization problems, whose solution sequence converges towards the equilibrium solution of the original problem. To achieve this goal they require differentiability of all functions involved, concavity of the industry revenue function and convexity of the firms' supply curves (to be defined later). Under the additional assumption of concavity of the inverse demand function, Szidarovsky and Yakowitz [6] suggest a somewhat simpler method, involving a binary search over the value of total output at equilibrium. Marcotte [3] has extended this method to cover the case where supply functions are not differentiable, and included in his analysis various algorithms for solving efficiently particular cases of the problem.

In the following we generalize the method of Szidarovsky and Yakowitz [6] to include the case where the demand functions are not necessarily concave or differentiable. However, we retain the assumption of concavity of the industry revenue function.

2. PROBLEM DEFINITION

In this section we follow the notation adopted by Murphy et al [4]. We define:

Inverse demand curve: $p(Q)$, defined for any output level $Q \geq 0$, gives the price that consumers are prepared to pay when quantity Q is available on the market.

Production level (output) of firm i: q_i, $i=1,2,\ldots,N$

Total output: $Q = \sum_{i=1}^{N} q_i$

Industry revenue curve: $Qp(Q)$

Cost function of firm i: $f_i(q_i)$, i=1,2,...,N, defined for $q_i \geq 0$.

Production vector: $q = (q_1,\ldots,q_N)$

Definition: A nonnegative production vector q* constitutes a Nash equilibrium if, for each firm i, q_i^* is solution of the maximization problem:

$$\max_{q_i \geq 0} \; q_i p(q_i + \sum_{j \neq i} q_j^*) - f_i(q_i) \qquad (1)$$

In the remainder of the paper we make the following assumptions about the functions $p(\cdot)$ and $f_i(\cdot)$:

Assumption A1. For each i the function $f_i(\cdot)$ is continuous and strictly convex on the nonnegative real axis.

Assumption A2. The inverse demand function is continuous and monotone decreasing for nonnegative Q.

Assumption A3. The industry revenue curve Qp(Q) is concave w.r.t. Q.

Assumption A4. Total output at equilibrium Q^* is positive.

Remark.

Depending on the structure of the problem, some assumptions could be weakened. In particular the strict convexity of the functions $f_i(\cdot)$ is only required to prove uniqueness of the total output at equilibrium Q^*. One could as well ask for strict concavity of the function Qp(Q). Even without such strict convexity or concavity assumptions, it is possible to prove the validity of the algorithm of section 4, at the expense of reduced readibility and more intricate conditions.

3. PRELIMINARY RESULTS

Define:

$$q_i^{min} = \arg \max_{q_i \geq 0} \{q_i p(q_i) - f_i(q_i)\} \tag{2}$$

which represents the monopolistic output of firm i. Also, let

$$q^{max} = (q_i^{max})_i = \arg \max_{\substack{q \geq 0 \\ Q}} \int_0^Q p(t)dt - \sum_{i=1}^N f_i(q_i) \tag{3}$$

$$\text{s.t. } Q = \sum_{i=1}^N q_i \quad .$$

q^{max} represents the output vector for a N-firm economy where marginal production costs equal consumption price whenever the production output is positive.

Assumption A1 through A5 ensure that (2) and (3) are strictly convex mathematical programming problems; thus q_i^{min} and q_i^{max} are unambiguously defined.

Set: $Q^{min} = \operatorname{Max}_i q_i^{min}$

and: $Q^{max} = \sum_{i=1}^N q_i^{max}$

we make the additional assumption:

Assumption A5. $Q^{max} < \infty$.

The following result shows that Q^* must be bounded below (respectively above) by Q^{min} (respectively Q^{max}).

Proposition 1. Under assumptions A1 through A5 we have:

$$0 < Q^{min} \leq Q^* \leq Q^{max} < \infty \ .$$

Proof 1. $Q^{min} > 0$.

Assumption A4 implies that at least one firm can make a profit in the oligopolistic market. The same firm can obviously make a profit in a monopolistic market. Therefore Q^{min} must be positive.

2. $Q^* \geq Q^{min}$.

Let the $^+$ and $^-$ signs stand for directional derivation form the right and left hand side respectively.

The optimality conditions of problem (1) give, for each firm i:

$$p(q_i^{min}) + q_i^{min}\, p^-(q_i^{min}) - f_i^-(q_i^{min}) \geq 0 \ . \tag{4}$$

Similarly we obtain, from optimization problem (1):

$$p(Q^*) + q_i^* p^+(Q^*) - f_i^+(q_i^*) \leq 0 \ . \tag{5}$$

Suppose $Q^* < q_i^{min}$ for some index i. Then:

$$\begin{aligned} p(q_i^{min}) + q_i^{min}\, p^-(q_i^{min}) - f_i^-(q_i^{min}) \\ < p(Q^*) + Q^*\, p^+(Q^*) - f_i^+(q_i^*) \\ \leq p(Q^*) + q_i^*\, p^+(Q^*) - f_i^+(q_i^*) \\ \leq 0 \qquad \text{by (5)} \end{aligned}$$

which contradicts (4). Therefore we must have:

$$Q^* \geq \operatorname*{Max}_i q_i^{min} = Q^{min} \ .$$

3. $Q^* \leq Q^{max}$.

Suppose $Q^* > Q^{max}$ and let i be such that $q_i^* > q_i^{max}$.
From the optimality conditions of (3) there comes:

$$p(Q^{max}) - f_i^+(q_i^{max}) \leq 0 \ .$$

We can write:

$$0 \le p(Q^*) + q_i^* p^-(Q^*) - f_i^-(q_i^*)$$

$$< p(Q^*) - f_i^+(q_i^{max}) \text{ since } p^-(Q^*) \text{ is nonpositive,}$$

$f_i^-(\cdot)$ is strictly increasing and $q_i^{max} < q_i^*$.

$$\le p(Q^{max}) - f_i^+(q_i^{max})$$

$$\le 0 \qquad \text{(contradiction)}$$

4. $Q^{max} < \infty$, by assumption A5. ∎

We now define a one-dimensional mapping whose fixed point will yield the total output at equilibrium Q^*. Knowledge of Q^* will in turn yield the values q^* of the corresponding equilibrium vectors.

Let $Q \in [Q^{min}, Q^{max}]$. Define, for each firm i, and $y_i \ge 0$, the quantity

$$q_i(y_i) = \arg \max_{q_i \ge 0} \{q_i p(q_i + y_i) - f_i(q_i)\} \tag{6}$$

$$\text{Let } S_i(Q) = \{q_i(y_i) \mid y_i + q_i(y_i) = Q \text{ for some } y_i(Q) \ge 0\} \tag{7}$$

and: $S(Q) = S_1(Q) \times S_2(Q) \times \ldots \times S_N(Q) \subset R^N$.

We construct the point-to-set mapping:

$$G: \; Q \to G(Q) = \{\tilde{Q} = \sum_{i=1}^{N} \tilde{q}_i \text{ where } \tilde{q}_i \in S_i(Q), \quad i = 1, 2, \ldots, N\} \subset R^N . \tag{8}$$

Proposition 2. Q^* is an equilibrium total output if and only if Q^* is a fixed point of the upper semicontinuous mapping G.

Proof. By definition of an equilibrium vector q^*, each q_i^* must be solution of problem (6) with $y_i = \sum_{j \neq i} q_j^*$. Therefore we have $Q^* \in G(Q^*)$. Conversely, if $Q^* \in G(Q^*)$, there exists a vector q^* such that: $Q^* = \sum_{i=1}^{N} q_i^*$. Such a vector is the required equilibrium.

To prove upper semicontinuity, consider the two sequences: $Q^n \underset{n\to\infty}{\longrightarrow} \bar{Q}$ and $q_i^n \underset{n\to\infty}{\longrightarrow} \bar{q}_i$, where $q_i^n \in S_i(Q^n)$. We want to show that $\bar{q}_i \in S_i(\bar{Q})$.

By definition of the set $S_i(Q)$ we have the inequality,

$$q_i^n p(Q^n) - f_i(q_i) \geq q_i p(q_i + Q^n - q_i^n) - f_i(q_i) \qquad \forall\, q_i \geq 0 .$$

By taking limits there comes:

$$\bar{q}_i p(\bar{Q}) - f_i(\bar{q}_i) \geq q_i p(q_i + \bar{Q} - \bar{q}) - f_i(\bar{q}_i) \qquad \forall\, q_i \geq 0$$

implying that $\bar{q}_i \in S_i(\bar{Q})$ ■(*)

Proposition 3. $Q \geq Q^{min}$ implies: $q_i(Q) \leq Q$.

Proof. Define, for each i, the point-to-set mapping:

$$Q_i : x \longrightarrow Q_i(x) = \{\arg\max_{0 \leq q_i \leq Q^{max}} q_i p(q_i + Q - x) - f_i(q_i)\}$$

for $x \in [0, Q^{max}]$.

(*) In the proof it is implicitly assumed that $S_i(Q)$ is nonempty for every nonnegative Q. This is a consequence of proposition 3 below, whose proof does not depend on the results of proposition 2, of course!

Assume that $p(\cdot)$ is zero on the negative axis. Although this extension will generally create a discontinuity for $p(\cdot)$ at 0, the function $Qp(Q)$ is still continuous at 0. This ensures that $Q_i(x)$ is well defined.

We then prove that Q_i is upper semicontinuous. To do this, let $x^n \underset{n\to\infty}{\longrightarrow} \bar{x}$ and $q_i^n \underset{n\to\infty}{\longrightarrow} \bar{q}_i$ where $q^n \in Q_i(x^n)$, $n=1,2,\ldots$

We have:

$$q_i^n p(q_i^n + Q - x^n) - f(q_i^n) \geq q_i p(q_i + Q - x^n) - f(q_i)$$

for all n and nonnegative q_i.

By taking limits: $\bar{q}_i p(\bar{q}_i + Q - \bar{x}) - f(\bar{q}_i) \geq q_i p(q_i - Q - \bar{x}) - f(q_i)$

for all nonnegative q_i.

Therefore: $\bar{q}_i \in Q_i(\bar{x})$

and $Q_i(x)$ is upper semicontinuous.

Now, $Q_i(x)$ must admit at least one fixed point, since it satisfies the hypothesis of Kakutani's theorem (see [1] for example). Any such point being a member of $S_i(Q)$ we can conclude that $S_i(Q)$ and $S(Q)$ are nonempty.

Suppose that $q_i(Q) > Q$. Then:

$$\begin{aligned} 0 &\leq p(Q) + q_i(Q)\ p^-(Q) - f_i^-(q_i(Q)) \\ &< p(Q) + Qp^+(Q) - f_i^+(Q) \\ &\leq p(q^{\min}) + q^{\min}\ p^+(q^{\min}) - f_i^+(q_i^{\min}) \quad \text{by assumption A3} \\ &< 0 \quad \text{(contradiction)} \end{aligned}$$

Therefore: $q_i(Q) \leq Q$, and: $y_i(Q) = Q - q_i(Q) \geq 0$. ∎

Proposition 4. The sets $S_i(Q)$ are closed and bounded intervals.

Proof 1. To prove that the sets $S_i(Q)$ are intervals it is sufficient to show that they are convex subsets of R. For ease of presentation we temporarily drop the subscript i.

Let q_a and q_b be members of S(Q) with q_a less than q_b. By definition of S(Q), q_a and q_b are solution of (6) with $y_a=Q-q_a$ and $y_b=Q-q_b$. Therefore:

$$q_a p(Q)-f(q_a) \geq qp(q+Q-q_a) - f(q) \quad \forall q \geq 0 \tag{9a}$$

$$q_b p(Q)-f(q_b) \geq qp(q+Q-q_b) - f(q) \quad \forall q \geq 0 \tag{9b}$$

Let $\lambda \in [0,1]$, $\bar{q} = \lambda q_a+(1-\lambda)q_b$ and $q \geq \bar{q}$. Then: $\bar{q} \geq q_a$.

We have:

$$\begin{aligned}\alpha &= qp(Q+q-\bar{q}) - f(q) - \bar{q}p(Q) + f(\bar{q}) \\ &= (q-\bar{q})\, p(Q+q-\bar{q}) + \bar{q}[p(Q+q-\bar{q})-p(Q)] - [f(q)-f(\bar{q})] \\ &\leq (q-\bar{q})\, p(Q+q-\bar{q}) + q_a[p(Q+q-\bar{q})-p(Q)] - [f(q)-f(\bar{q})].\end{aligned}$$

The last inequality follows from: $q_a \leq \bar{q}$ and $p(Q+q-\bar{q}) \leq p(Q)$. Also, convexity of f(q) implies:

$$f(q)-f(\bar{q}) \geq f(q_a+q-\bar{q}) - f(q_a)$$

Thus: $$\begin{aligned}\alpha &\leq (q-\bar{q})\, p(Q+q-\bar{q}) + q_a[p(Q+q-\bar{q}) - p(Q)] - f(q_a+q-\bar{q}) + f(q_a) \\ &= (q_a+q-\bar{q})\, p(Q+q-\bar{q}) - f(q_a+q-\bar{q}) - q_a p(Q) + f(q_a) \\ &\leq 0 \quad \text{by (9a).}\end{aligned}$$

We deduce: $qp(Q+q-\bar{q}) - f(q) \leq \bar{q}p(Q) - f(\bar{q}) \quad \forall q \geq \bar{q}.$ (10)

Similarly we can show, using (9b), that we have:

$$qp(Q+q-\bar{q}) - f(q) \leq \bar{q}p(Q) - f(\bar{q}) \qquad \forall q \leq \bar{q} \ (q \geq 0). \tag{11}$$

From (10) and (11) we conclude that $\bar{q} \in S(Q)$. Therefore $S(Q)$ is convex. It is worth noting that this result does not depend in any manner on the concavity of the industry revenue curve $Qp(Q)$.

2. $S(Q)$ is closed.

Let $\{q_n | q_n \in S(Q)\}$ be a converging sequence of elements of $S(Q)$, that is to say: $q^n \underset{n\to\infty}{\longrightarrow} \bar{q}$.

We want to show that $\bar{q}$ is in $S(Q)$.

We can write:

$$q^n p(Q) - f(q^n) \geq qp(Q+q-q^n) - f(q) \qquad \forall n, \forall q \geq 0.$$

By taking limits on both sides of the inequality we find:

$$\bar{q}p(Q) - f(\bar{q}) \geq qp(Q+q-\bar{q}) - f(q) \qquad \forall q \geq 0$$

Thus: $\bar{q}$ is in $S(Q)$ and $S(Q)$ is closed.

3. Finally, $S(Q)$ is obviously bounded since $S(Q) \subseteq [0, Q^{max}]$. This completes the proof. ∎

Definition: $\{q_i(Q)\} = [G_i^{min}(Q), G_i^{max}(Q)]$.

Proposition 5. $\bar{Q} > Q \geq Q^{min}$ implies that $\dfrac{G_i^{max}(\bar{Q})}{\bar{Q}} < \dfrac{G_i^{min}(Q)}{Q}$, i.e. the "interval function" $\dfrac{G(Q)}{Q}$ is strictly decreasing.

Proof. By definition of $q_i(q)$ and $y_i(Q)$ we have:

$$p(Q) + q_i(Q)\, p^+(Q) - f_i^+(q_i(Q)) \leq 0 \ . \tag{12}$$

Set $\alpha = \dfrac{q_i(Q)}{Q}$ and write (12) as:

$$p(Q) + \alpha Q p^+(Q) - f^+(\alpha Q) \leq 0 \tag{13}$$

or: $$\alpha \geq \frac{f^+(\alpha Q) - p(Q)}{Q p^+(Q)}$$

Then: $$\frac{1}{\alpha} \leq \frac{-Q p^+(Q)}{p(Q) - f^+(\alpha Q)}$$

$$\frac{1}{\alpha} - 1 \leq \frac{-Q p^+(Q) - p(Q) + f^+(\alpha Q)}{p(Q) - f^+(\alpha Q)}$$

Finally: $$\frac{1}{\alpha} + \frac{[Qp(Q)]^+ - f^+(\alpha Q)}{p(Q) - f^+(\alpha Q)} \leq 1$$

Similar calculations yield:

$$\frac{1}{\alpha} + \frac{[Qp(Q)]^- - f^-(\alpha Q)}{p(Q) - f^-(\alpha Q)} \geq 1$$

Now let $\bar{Q} > Q \geq Q^{min}$ and suppose that

$$\bar{\alpha} = \frac{q_i(\bar{Q})}{\bar{Q}} \geq \alpha \ .$$

By proposition 3, it follows that α is less or equal to 1. This, combined with (13) yields:

$$p(Q) + Qp^{+}(Q) - f^{+}(\alpha Q) \leq 0$$

Also: $[\bar{Q}p(\bar{Q})]^{-} - f^{-}(\alpha\bar{Q}) < p(Q) + Qp^{+}(Q) - f^{+}(\alpha Q) \leq 0$.

But: $p(\bar{Q}) + \bar{\alpha}\bar{Q}p^{-}(\bar{Q}) - f^{-}(\bar{\alpha}\bar{Q}) \geq 0$.

Thus: $$1 \leq \frac{1}{\alpha} + \frac{[\bar{Q}p(\bar{Q})]^{-} - f^{-}(\bar{\alpha}\bar{Q})}{p(\bar{Q}) - f(\bar{\alpha}\bar{Q})}$$

$$< \frac{1}{\alpha} + \frac{[Qp(Q)]^{+} - f^{+}(\alpha Q)}{p(Q) - f^{+}(\alpha Q)}$$

≤ 1 (contradiction)

Thus: $\frac{q_i(Q)}{Q}$ is a decreasing function of Q. ∎

Proposition 6. The equilibrium total output Q^{*} is unique.

Proof. This a direct consequence of the statement of proposition 5. ∎

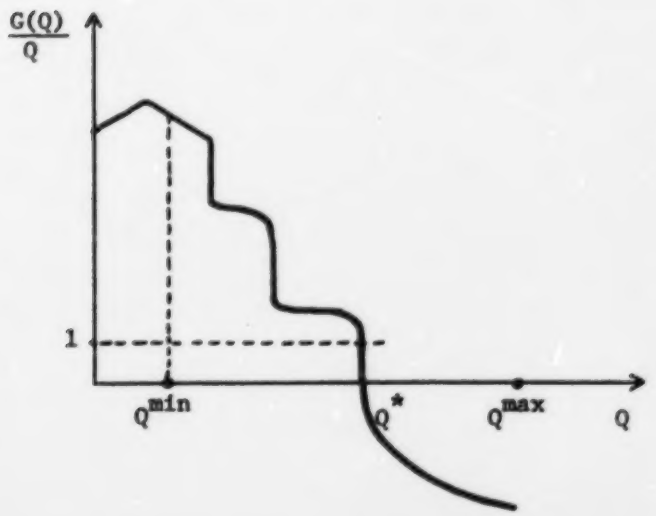

Figure 1

Proposition 7. If $p(\cdot)$ and $f_i(\cdot)$, $i=1,\ldots,N$ are differentiable at Q^*, then the equilibrium output vector q^* is unique.

Proof. Under the differentiability hypothesis, q^* satisfies:

$$p(Q^*) + q_i^* \, p'(Q^*) - f_i'(q_i^*) \leq 0$$

$$q_i^*[p(Q^*) + q_i^* \, p'(Q^*) - f_i'(q_i^*)] = 0$$

Since $f_i'(\cdot)$ is strictly increasing, there is at most one solution to the equation:

$$p(Q^*) + q_i \, p'(Q^*) - f_i'(q_i) = 0 \qquad (14)$$

for fixed Q^*. ∎

4. A BINARY SEARCH ALGORITHM

Proposition 5 shows that the point-to-set mapping $G(\cdot)$ can be written:

$$G(Q) = [G^{\min}(Q), \, G^{\max}(Q)]$$

with $$G^{\min}(Q) = \sum_{i=1}^{N} G_i^{\min}(Q) \quad \text{and} \quad G^{\max}(Q) = \sum_{i=1}^{N} G_i^{\max}(Q).$$

The equilibrium total output Q^* satisfies: $1 \in \frac{G(Q^*)}{Q^*}$ and can be located by performing a binary search on either $Q-G(Q)$ or $\frac{G(Q)}{Q} - 1$. The corresponding solution set is the intersection of the hyper-rectangle $S(Q^*)$ with the hyperplane

$$H = \{q \mid \sum_{i=1}^{N} q_i = Q^*\}.$$

Such a point Q^* must exist since the mapping is uppersemicontinuous (see proposition 2). Furthermore it is unique by proposition 6.

However, as shown by an example in [3], this does not imply that the vector q^* corresponding to Q^* is unique.

Initial lower and upper bounds on Q can be taken as Q^{min} and Q^{max} respectively. We then have the obvious relation:

$$\frac{G(Q^{min})}{Q^{min}} \geq 1 \geq \frac{G(Q^{max})}{Q^{max}} \quad .$$

In the special case where all functions involved are continuously differentiable, any positive $q_i(Q)$ is the unique root of the equation:

$$p(Q) + q_i\, p'(Q) - f_i'(q_i) = 0 \ .$$

This root can itself be found by binary search, or more sophisticated procedures using higher derivatives information, provided such information exists.

It is worth mentioning that a similar algorithm has been proposed by Szidarovsky and Yakowitz [6] in the case where $p(\cdot)$ is concave, and $f(\cdot)$ and $p(\cdot)$ are differentiable. They showed the validity of their approach by proving that $Q - G(Q)$ is a continuously increasing function of Q. Our approach can be considered a validation of this approach in a more general setting.

The binary search algorithm can be implemented under less stringent conditions than those adopted so far. It is sufficient that proposition 4 holds; therefore we can do without assumption A3. However, in this instance, uniqueness of the quantity Q^* is not guaranteed.

5. NUMERICAL EXAMPLE

In this section we simply apply our algorithm to the example of Murphy et al [6] (see appendix II). The cost functions are given by:

$$f_i(q_i) = c_i q_i + \frac{\beta_i}{\beta_i+1} 5^{\frac{-1}{\beta_i}} \cdot q_i^{\frac{\beta_i+1}{\beta_i}} \qquad i=1,2,\ldots N \tag{11}$$

Also we have the inverse demand function:

$$p(Q) = 5000^{\frac{1}{1.1}} Q^{-\frac{1}{1.1}} \tag{12}$$

Coefficient values are given below for the five firms involved:

c's :	10	8	6	4	2
β's :	1.2	1.1	1.0	.9	.8

For the first two iterations, straight bisection is applied. Starting from the third iteration, the method of the false position is adopted and performs remarkably well, indicating that the function is almost linear between the lower and upper bounds involved.

Starting with the same bounds as Murphy et al, the algorithm provided a solution as good as theirs in four iterations (compared to six). After six iterations, the q_i' s are correct to five significant digits. Calculations were performed on Radio Shack programmable calculator EC 4004 and the results are shown in table 1.

Murphy et al's solution is shown in table 2. The true value for Q^* lies in the interval [204.2954, 204.2955].

TABLE 1: Numerical results on Murphy et al's example

Iteration	Number of significant digits retained	Q_{min}	Q_{max}	Q	q_1	q_2	q_3	q_4	q_5	Q-G(Q) (*)
1	3	40.0	235.4	137.7	52.6	54.5	54.2	51.4	46.3	121.3
2	3	137.7	235.4	186.5	41.6	45.6	46.7	45.1	41.1	-33.6
3	3	186.5	235.4	210.9	35.2	40.4	42.6	41.8	38.5	12.4
4	4	186.5	210.9	204.32	36.92	41.81	43.70	42.65	39.18	.06
6	5	204.288	204.320	204.294	36.933	41.818	43.707	42.659	39.179	-.002
9	7	204.2953	204.320	204.2954	36.93252	41.81815	43.70658	42.65924	39.17895	-.00004

(*) Search performed on the quantity Q-G(Q).

TABLE 2: Solution found by Murphy et al after 6 iterations

Q	q_1	q_2	q_3	q_4	q_5
204.306	36.912	41.842	43.705	42.665	39.182

6. CONCLUSION

We have presented an algorithm for solving oligopolistic market equilibrium problems that covers the case where supply and demand functions are nondifferentiable. Furthermore, specialisation of the algorithm to the differentiable case yields a procedure that is at least as efficient as procedures previously proposed in the litterature.

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APPENDIX I

To show the relevance of nondifferentiable situations, we consider an example of oligopolistic derived demand whose formulation is due to Breton and Haurie [2].

Consider the single commodity network of figure 2.

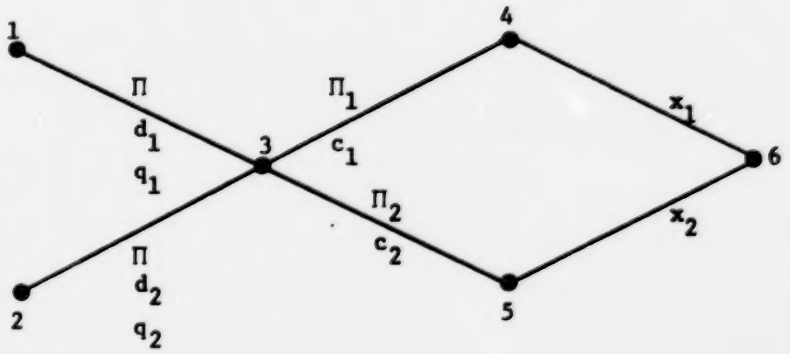

Figure 2

Nodes 1 and 2 represent extractors of the raw product; nodes 3, 4 and 5 represent transformation firms and node 6 the final demand market.

Oligopolists at a given level behave among themselves according to the Cournot principle; they integrate into their profit maximisation scheme the reaction functions of the nodes situated to their right. Price discrimination can occur. Let:

Π : unique selling price to node 3

Π_1: selling price from 3 to 4

Π_2 selling price from 3 to 5.

d_1, d_2, c_1, c_2 : unit transportation costs

q_1, q_2, x_1, x_2 : flows of commodity on the network

$p(Q) = \alpha - Q$: inverse demand function.

For duopolists 4 and 5, the problems are, respectively:

$$\underset{x_1 \geq 0}{\text{Max}} \; x_1(\alpha - x_1 - x_2) - x_1(\Pi_1 + c_1)$$

$$\underset{x_2 \geq 0}{\text{Max}} \; x_2(\alpha - x_1 - x_2) - x_2(\Pi_2 + c_2).$$

Let $x_1(\Pi_1, \Pi_2)$ and $x_2(\Pi_1, \Pi_2)$ be the equilibrium solution of the above system. The monopolist problem (node 3) is then:

$$\underset{\Pi_1, \Pi_2}{\text{Max}} \; (\Pi_1 - \Pi)\, x_1(\Pi_1, \Pi_2) + (\Pi_2 - \Pi)\, x_2(\Pi_1, \Pi_2) \; .$$

Some simple algebra yields the derived inverse demand function at node 1, as a function of total quantity produced $Q=q_1+q_2$:

$$\Pi(Q) = \begin{cases} (\alpha-c_1)-4Q & 0 \le Q \le \frac{c_2-c_1}{2} \\ \frac{2\alpha-c_1-c_2}{2} - 3Q & \frac{c_2-c_1}{2} \le Q \le \frac{2\alpha-c_1-c_2}{6} \\ 0 & Q \ge \frac{2-c_1-c_2}{6} \end{cases} \qquad (10)$$

This function is convex and nondifferentiable at the point $\frac{c_2-c_1}{2}$.

APPENDIX II

Consider the concave mathematical programming problem:

$$\underset{Q,q\geq 0}{\text{Max}} \; p(Q) \sum_{i=1}^{N} q_i + \frac{1}{2} p'(Q) \sum_{i=1}^{N} q_i^2 - \sum_{i=1}^{N} f_i(q_i)$$

$$\text{s.t.} \quad \sum_{i=1}^{N} q_i = Q$$

and let $\lambda(Q)$ be the optimal Lagrange multiplier (dual variable) associated with the constraint. Then we have:

Theorem (Murphy, Sherali, Soyster).

Under assumptions A1 through A5 and differentiability there holds:

1. $\lambda(Q)$ is continuous and strictly increasing in Q.
2. $\lambda(Q) = 0$ if and only if $Q = Q^*$. ∎

The algorithm proposed searches the root Q^* of the equation $\lambda(Q) = 0$ using a binary search technique.